II Fibrations

A Locally Trivial Fibrations

a fiber bundle (or locally trivial fibration or fibration) is a 4-tuple (E, B, F, p) where

E, B, F are topological spaces and
$p: E \to B$ is a continuous map

such that for all $x \in B$, $\exists$ an open set $U \subset B$

and a homeomorphism $\phi: p^{-1}(U) \to U \times F$
s.t. $\pi_1 \circ \phi = p$

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\phi"] \arrow[dr, "p"'] & & U \times F \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

projection to first factor

denote this
$$\begin{tikzcd} F \arrow[r] & E \arrow[d, "p"] \\ & B \end{tikzcd}$$

E is called the total space
B " " base space
F " " fiber
p " " projection

$\phi: p^{-1}(U) \to U \times \mathbb{R}$ is called a local trivialization

examples: 1) $E = B \times F$ a product space

2) Möbius band

$$M = \mathbb{R}\times\mathbb{R} /_{(x,y)\sim(x+1,-y)}$$ let $q:\mathbb{R}\times\mathbb{R}\to M$ be the quotient map

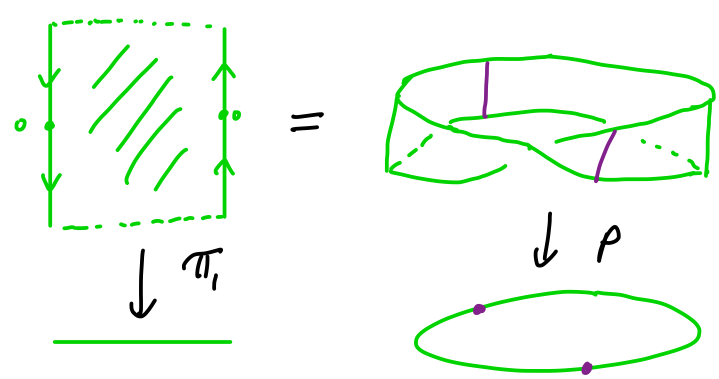
<u>note</u>: every $(x,y)\in\mathbb{R}\times\mathbb{R}$ in M is equivalent to a point in $[0,1]\times\mathbb{R}$

there are no identifications in $(0,1)\times\mathbb{R}$

but $\{1\}\times\mathbb{R}$ is identified with $\{0\}\times\mathbb{R}$ by $y\mapsto -y$

now $\pi_1:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ induces a map

$$p: M\to S^1 = \mathbb{R}/_{x\mapsto x+1}$$



given $x\in S^1$ if it is "in" $(0,1)$ then consider

$q|_{(0,1)\times\mathbb{R}}:(0,1)\times\mathbb{R}\to M$ is an embedding

let $\phi: \underbrace{q((0,1)\times\mathbb{R})}_{p^{-1}((0,1))}\to(0,1)\times\mathbb{R}$ be its inverse so $\begin{tikzcd} p^{-1}(0,1) \arrow[r, "\Phi"] \arrow[dr, "p"'] & (0,1)\times\mathbb{R} \arrow[d, "\pi_1"] \\ & (0,1) \end{tikzcd}$ is a local trivialization

similarly if $x \in S^1$ is "in" $(1/2, 3/2)$ can use

$$q|_{(1/2,3/2)\times\mathbb{R}}$$

so $p: M \to S^1$ is a fiber bundle

3) S^{2n-1} = unit sphere in $\mathbb{C}^n$

recall $S^1 \subset \mathbb{C}$ (unit circle) acts on S^{2n-1}

by $S^1 \times S^{2n-1} \to S^{2n-1} : (\lambda, (z_1, \dots z_n)) = (\lambda z_1, \dots \lambda z_n)$

exercise/recall: $S^{2n-1}/S^1 \cong \mathbb{C}P^{n-1}$

exercise: show $\begin{tikzcd} S^1 \arrow[r] & S^{2n-1} \arrow[d] \\ & \mathbb{C}P^n \end{tikzcd}$ is a fiber bundle

4) in general if G is a Lie group (recall this means G is a smooth manifold and a group such that products and inverses are smooth maps)

and H is a compact subgroup of G

then
$$\begin{tikzcd} H \arrow[r] & G \arrow[d] \\ & G/H \end{tikzcd}$$
is a fiber bundle

exercise: prove this

invertable $n \times n$ matrices with $\mathbb{R}$ entries

std inner product

e.g. recall $O(n) = \{ A \in GL(n, \mathbb{R}) : \langle Ax, Ay \rangle = \langle x, y \rangle \}$

orthogonal group

$= \{ A \in GL(n, \mathbb{R}) : A^T = A^{-1} \}$

special orthogonal group

$SO(n) = \{ A \in O(n) : \det A = 1 \}$

recall: from differential topology we know $O(n)$ and $SO(n)$ have dimension $\frac{n(n-1)}{2}$ and $O(n)$ has two components with $SO(n)$ the identity cpt.

exercise: $SO(1) = \{1\}$

$SO(2) \cong S^1$

$SO(3) \cong \mathbb{RP}^3$

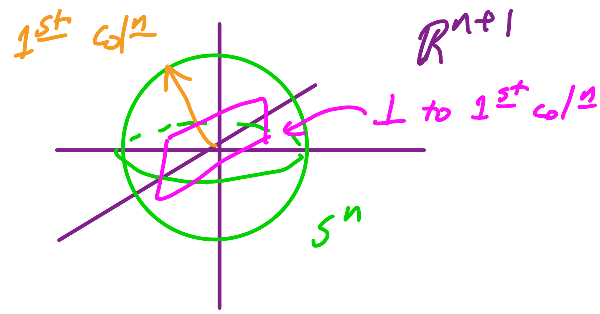
a)
$$\begin{tikzcd} SO(n) \arrow[r] & SO(n+1) \arrow[d] \\ & SO(n+1)/SO(n) \cong S^n \end{tikzcd}$$

here $SO(n) < SO(n+1)$

$$A \longmapsto \begin{pmatrix} 1 & 0 \cdots 0 \\ 0 & \\ \vdots & A \\ 0 & \end{pmatrix}$$

exercise: prove this

hint: note first column of $B \in SO(n+1)$ is an elt of S^n



b) let $V_{n,k}$ = orthogonal k-frames in $\mathbb{R}^n$ ← called the Steifel mfd

exercise: $V_{n,k} = O(n)/O(n-k)$

so $O(n-k) \longrightarrow O(n)$
$\qquad\qquad\qquad\qquad\downarrow$ is a fiber bundle
$\qquad\qquad\qquad\quad V_{n,k}$

note this implies

$$V_{n,n} \cong O(n)$$
$$V_{n,1} \cong S^{n-1}$$
$$V_{n,n-1} \cong SO(n)$$

for $k < n$ can also show $V_{n,k} \cong SO(n)/SO(n-k)$

c) $G_{n,k}$ = k-dimensional planes in $\mathbb{R}^n$ Grassmannian

exercise: $G_{n,k} = O(n) / O(k) \times O(n-k)$

5) recall $U(n) = \{ A \in GL(n;\mathbb{C}) : \langle Av, Au \rangle = \langle v, u \rangle \}$

invertable matricies

preserves Hermition inner product

$\bar{v} \cdot u$

unitary group

$= \{ A \in GL(n;\mathbb{C}) : \bar{A}^T = A^{-1} \}$

special unitary group $SU(n) = \{ A \in U(n) : \det A = 1 \}$

recall: from differential topology we know $U(n)$ is a manifold of dimension n^2 and $SU(n)$ of dimension $n^2 - 1$

$$\begin{array}{ccc} SU(n) & \to & U(n) \\ & & \downarrow \\ & & S^1 \end{array}$$

is a bundle

exercise: $U(1) \cong S^1$

$SU(2) = S^3$

$U(2) = S^3 \times S^1$

a) $$\begin{array}{ccc} SU(n) & \to & SU(n+1) \\ & & \downarrow \\ & & SU(n+1)/SU(n) \cong S^{2n+1} \end{array}$$

where $SU(n) \longrightarrow SU(n+1)$

$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

exercise: prove this

b) let $V_{n,k}(\mathbb{C})$ = orthogonal k-frames in $\mathbb{C}^n$

exercise: $V_{n,k}(\mathbb{C}) = U(n)/U(n-k)$

c) $G_{n,k}(\mathbb{C})$ = k-dimensional planes in $\mathbb{R}^n$

exercise: $G_{n,k}(\mathbb{C}) = U(n)/U(k) \times U(n-k)$

6) if $f: M \to N$ is a smooth map such that

i) f is surjective

ii) f is a submersion

iii) f is proper (automatic if M compact)

preimage of compact is compact

then
$$\begin{array}{ccc} f^{-1}(p) & \longrightarrow & M \\ & & \big\downarrow f \\ & & N \end{array}$$
is a fiber bundle for any $p \in N$

this is Ehresmann's lemma

7) Vector bundles are fiber bundles with fiber $\mathbb{R}^k$ or $\mathbb{C}^k$ (with extra "structure", see later in notes)

e.g. a) $\begin{array}{c} TM \\ \downarrow \\ M \end{array}$ b) $\begin{array}{c} T^*M \\ \downarrow \\ M \end{array}$ c) $N^n \subset M^m$ a submfd

$\nu(N)$ = normal bundle

$$\begin{array}{ccc} \mathbb{R}^{m-n} & \to & \nu(N) \\ & & \downarrow \\ & & N \end{array}$$

8) covering spaces $\begin{array}{c} \tilde{M} \\ \downarrow \\ M \end{array}$ is a bundle with discrete fiber

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given a fiber bundle $E \xrightarrow{p} B$ and a map $f: A \to B$

the pull-back of E to A is

$$f^*E = \{(a,e) \in A \times E : f(a) = p(e)\}$$

define $p: f^*E \to A : (a,e) \longmapsto a$

exercise: 1) show $f^*E \to A$ is a fiber bundle with same fiber as $E \xrightarrow{p} B$

2) if A is a subset of B and $f: A \to B$ is inclusion, then show $f^*E = E|_A$ i.e. $E|_A = p^{-1}(A)$

3) $\tilde{f}: f^*E \to E : (a,e) \to e$ is a bundle map

4) if $E = B \times F$ then $f^*E \cong A \times F$

Hint: f^*E is $\Gamma \times F$ where Γ is the graph of f in $A \times B$ and $\Gamma \cong A$

if $E \xrightarrow{p} B$ and $E' \xrightarrow{p'} B$ are bundles we say they are <u>bundle isomorphic</u> if $\exists$ a homeomorphism $h: E \to E'$ such that

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & & E' \arrow[dl, "p'"] \\ & B \arrow[u, phantom, "\circ"] & \end{tikzcd}$$

commutes

denote this $E \cong E'$

Th$^{\underline{m}}$ 1:

> if $f_i : A \to B$, $i = 0,1$, are homotopic and A is a CW complex, ← can weaken
>
> then $f_0^* E \cong f_1^* E$

this is a corollary of

Th$^{\underline{m}}$ 2 (Covering Homotopy Property)

> let $p_0 : E \to B$ and $q : Z \to Y$ be fiber bundles with the same fiber F

suppose B is normal and locally compact

given $\tilde{h}_0 : E \to Z$ and $h_0 : B \to Y$ such that

$$\begin{tikzcd} E \arrow[r, "\tilde{h}_0"] \arrow[d, "p_0"'] & Z \arrow[d, "q"] \\ B \arrow[r, "h_0"] & Y \end{tikzcd}$$

commute (called a <u>bundle map</u>)

and $H : B \times [0,1] \to Y$ a homotopy of h_0

then $\exists$ a homotopy $\tilde{H} : E \times [0,1] \to Z$ of bundle maps covering H

<u>Proof of Th$^{\underline{m}}$ 2</u>:

we assume B is compact (and leave the locally compact case as an exercise)

<u>Idea</u>: break Z into pieces where the bundle is trivial $U \times F$

th$^{\underline{m}}$ is clear hear,

then we put the homotopies together

<u>details</u>: let $\{V_\beta\}$ be a cover of Y by locally trivial charts, so we have

$$q^{-1}(V_\beta) \xrightarrow[\cong]{\phi_\beta} V_\beta \times F$$

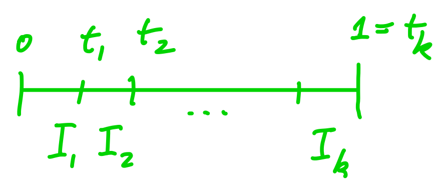
$\{H^{-1}(V_\beta)\}$ is an open cover of $B \times [0,1]$

since B is compact we have a finite subcover, and in particular we have a finite number of open sets $\{U_\alpha \times I_j\}$ covering $B \times [0,1]$

s.t. $H(U_\alpha \times I_j) \subset V_\beta$ some β

(note H^*E is trivial over $U_\alpha \times I_j$ by exercise above)

here we can take the I_j



we inductively assume we have constructed the lift $\tilde{H} : E \times [0, t_\ell] \to Z$

and extend to $E \times [0, t_{\ell+1}]$ (note: $E \times \{0\}$ done)

for each $x \in B$ $\exists$ nbhds W, W' s.t.

$$x \in W \subset \overline{W} \subset W'$$

here we use normal

and $\overline{W}' \subset U_i$ for some i

choose a finite number of the $\{W_i, W_i'\}_{i=1}^{s}$ s.t. the $\{W_i\}$ cover B

Urysohn's lemma $\Rightarrow \exists$ maps

$$u_i : B \to [t_\ell, t_{\ell+1}]$$

s.t. $u_i(\overline{W}_i) = t_{\ell+1}$ and $u_i(B - W_i') = t_\ell$

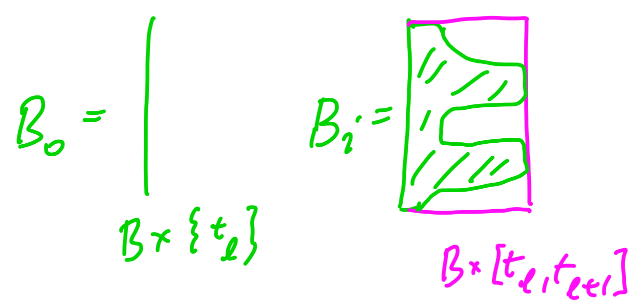
set $\tau_0(x) = t_\ell$ and

$$\tau_i(x) = \max\{u_1(x), \dots, u_i(x)\}$$

note $t_\ell = \tau_0(x) \le \tau_1(x) \le \dots \le \tau_s(x) = t_{\ell+1}$

set $B_i = \{(x,t) \in B \times [0,1] : t_\ell \le t \le \tau_i(x)\}$

so



$B_s = B \times [t_\ell, t_{\ell+1}]$

and E_i to be the part of $E \times [0,1]$ over B_i

so $E \times \{t_\ell\} = E_0 \subseteq E_1 \subseteq \dots \subseteq E_s = E \times [t_\ell, t_{\ell+1}]$

we have assumed $\tilde{H}$ is defined on $E \times [0, t_\ell]$

so it is defined on $E \times \{t_\ell\} = E_0$

we now inductively extend $\tilde{H}$ over E_i

note: if $(x,t) \in B_i - B_{i-1}$ then

$$\tau_{i-1}(x) < t \leq \tau_i(x)$$

so $u_i(x) > \tau_{i-1}(x)$

$\therefore (x,t) \in W_i' \times [t_\ell, t_{\ell+1}]$

by def$^{\underline{n}}$ $W_i' \times [t_\ell, t_{\ell+1}] \subset U_\alpha \times I_k$

so $H(B_i - B_{i-1}) \subset V_\beta$ some β

where $q^{-1}(V_\beta) \xrightarrow{\phi_\beta} V_\beta \times F$

let $\rho_\beta : q^{-1}(V_\beta) \longrightarrow F$ be ϕ_β composed with projection

now for $(e,t) \in E_i - E_{i-1}$ with $p(e) = x \in B$

set $\tilde{H}(e,t) = \phi_\alpha^{-1}(H(x,t), \rho_\beta(\tilde{H}\overbrace{(e, \tau_{i-1}(x))}^{E_{i-1}}))$

exercise: check this extends $\tilde{H}$ ▦

Proof of Th$^{\underline{m}}$ 1:

let $f_1 : A \to B$ be as in statement of th$^{\underline{m}}$

$H : A \times [0,1] \to B$ a homotopy f_0 to f_1

now
$$\begin{tikzcd} f_0^*E \arrow[r, "\tilde{f}_0"] \arrow[d] & E \arrow[d, "p"] \\ A \arrow[r, "f_0"] & B \end{tikzcd}$$
is a bundle map

so by Th$\underline{\text{m}}$ 2 $\exists$ a homotopy $\tilde{H}$

$$\begin{tikzcd} f_0^*(E) \times [0,1] \arrow[r, "\tilde{H}"] \arrow[d] & E \arrow[d, "p"] \\ A \times [0,1] \arrow[r, "H"] & B \end{tikzcd}$$

this induces a map

$$\begin{tikzcd} f_0^*(E) \times [0,1] \arrow[r, "\bar{H}"] \arrow[d] & H^*(E) \arrow[d] \\ A \times [0,1] \arrow[r, "id"] & A \times [0,1] \end{tikzcd}$$

$\{(x,e) \in A \times E : f_0(x) = p(e)\}$

$= \{(x,t,e) \in A \times [0,1] \times E : H(x,t) = p(e)\}$

where $\bar{H}((x,e),t) = (x,t,\tilde{H}(x,e,t))$

note $\bar{H}$ is a bundle isomorphism
since it induces identity on base
and all fibers

restricting $\bar{H}$ to $f_0^*(E) \times \{1\}$ gives a
bundle isomorphism
$$\begin{tikzcd} f_0^*(E) \arrow[r, "\bar{H}"] \arrow[d] & f_1^*(E) \arrow[d] \\ A \times \{1\} \arrow[r, "id"] & A \times \{1\} \end{tikzcd}$$

Corollary 3:

$\boxed{\text{If } X \text{ is contractible (and locally compact and normal), then any fiber bundle } E \to X \text{ is trivial } E \cong X \times F}$

Proof: X contractible means the identity map $f_0 : X \to X$ is homotopic to a constant map $f_1 : X \to X$

$$\therefore \; f_0^*(E) = E \cong f_1^*(E) = X \times F$$

exercise   exercise

B Serre Fibrations

a continuous map $p : E \to B$ is called a fibration (or a Serre fibration) if it has the homotopy lifting property (HLP): ← for this sometimes require Y to be CW complex

given a function $\tilde{g} : Y \to E$ and a homotopy $G : Y \times [0,1] \to B$ of $p \circ \tilde{g}$ (i.e. $G(y,0) = p \circ \tilde{g}(y)$)

then $\exists$ a homotopy $\tilde{G} : Y \times [0,1] \to E$

such that $p \circ \tilde{G} = G$

$$\begin{tikzcd} Y \times \{0\} \arrow[r, "\tilde{g}"] \arrow[d] & E \arrow[d, "p"] \\ Y \times [0,1] \arrow[r, "G"] \arrow[ur, dashed, "\tilde{G}"] & B \end{tikzcd}$$

note: Th$^{\underline{m}}$ 2 says a locally trivial fibration is a fibration

indeed: note $Y \to Y : y \mapsto y$ is a bundle with fiber $\{pt\}$, so Th$^{\underline{m}}$ 2 $\Rightarrow$ HLP

exercise: If $p: E \to B$ is a fibration and $f: X \to B$ is continuous, then show $p_f : f^*E \longrightarrow X$ is a fibration where

$$f^*E = \{(x,e) \in X \times E : p(x) = f(e)\}$$

called the pull-back bundle

and

$$p_f(x,e) = x$$

Th$^{\underline{m}}$ 4: If $p: E \to B$ is a Serre fibration and $x_0, x_1 \in B$ are in the same path component, then $p^{-1}(x_0)$ and $p^{-1}(x_1)$ are homotopy equivalent

(so upto homotopy fibrations have fibers)

Proof: let $F_i = p^{-1}(x_i)$
and γ a path from x_0 to x_1

$$\begin{tikzcd} F_0 \arrow[r, hook, "i"] \arrow[d] & E \arrow[d, "p"] \\ F_0 \times [0,1] \arrow[r] & B \end{tikzcd}$$
$$(x,t) \longmapsto \gamma(t)$$

homotopy of $p \circ i$

so we can lift to get a homotopy

$$F_0 \times [0,1] \xrightarrow{A^\gamma} E$$

and $A_1^\gamma : F_0 \longrightarrow F_1$ is a continuous map

Claim: if γ_0 and γ_1 are homotopic rel end points then A^{γ_0} and A^{γ_1} are homotopic and hence $A_1^{\gamma_0} \simeq A^{\gamma_1}$

assume the claim for now, then

$$A_1^\gamma : F_0 \longrightarrow F_1$$
$$A_1^{\gamma^{-1}} : F_1 \longrightarrow F_0$$

$A_1^\gamma \circ A_1^{\gamma^{-1}}$ comes from lifting $\alpha^{-1} * \alpha$

so $A_1^\gamma \circ A_1^{\gamma^{-1}} \simeq A^{const} = id_{F_1}$

similarly $A_1^{\gamma^{-1}} \circ A_1 \simeq id_{F_0}$

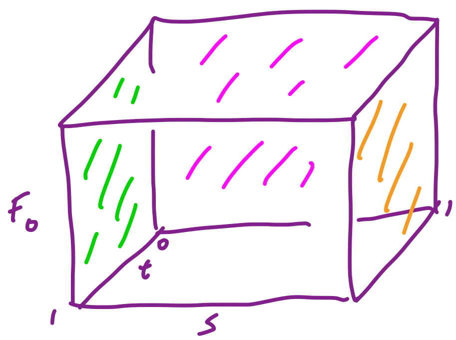
$\therefore F_0 \simeq F_1$

Proof of Claim: let $H : [0,1] \times [0,1] \to B$ be the homotopy γ_0 to γ_1

Consider

$$\Lambda : F_0 \times [0,1] \times [0,1] \to B : (e, s, t) \mapsto H(s,t)$$

we want to lift to E



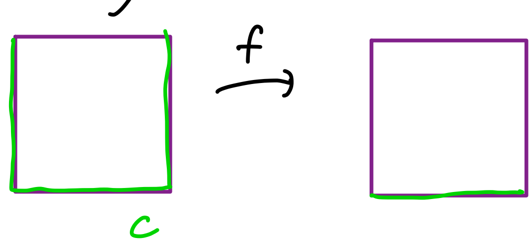
on $F_0 \times [0,1] \times \{0\} \to E$ by $\underline{A^{\gamma_0}}$

on $F_0 \times [0,1] \times \{1\} \to E$ by $\underline{A^{\gamma_1}}$ } call G

on $F_0 \times \{0\} \times [0,1] \to E : \underline{(e, 0, s) \mapsto e}$

set $C = ([0,1] \times \{0,1\}) \cup (\{0\} \times [0,1]) \subset [0,1] \times [0,1]$

$\exists$ a homeomorphism $f : [0,1] \times [0,1] \to [0,1] \times [0,1]$ taking C to $[0,1] \times \{0\}$



now we see

$$\begin{tikzcd}
F_0 \times [0,1] \times \{0\} \arrow[d, "i"] & F_0 \times C \arrow[l, "id_{F_0} \times f"'] \arrow[r, "G"] \arrow[d, "i"] & E \arrow[d, "p"] \\
F_0 \times [0,1] \times [0,1] & F_0 \times [0,1] \times [0,1] \arrow[l, "id_{F_0} \times f"] \arrow[r, "\Lambda"] & B
\end{tikzcd}$$

note $id_{F_0} \times f$ is a homeomorphism and so

we get
$$\begin{tikzcd}
F_0 \times [0,1] \times \{0\} \arrow[r] \arrow[d] & E \arrow[d, "p"] \\
F_0 \times [0,1] \times [0,1] \arrow[r] \arrow[ur, dashed] & B
\end{tikzcd}$$

so the HLP say there is a lift

composing with $id_{F_0} \times f$ we get
$$\begin{tikzcd}
F_0 \times C \arrow[r, "G"] \arrow[d, "i"'] & E \arrow[d, "p"] \\
F_0 \times [0,1] \times [0,1] \arrow[r, "\Lambda"] \arrow[ur, dashed, "\tilde{\Lambda}"] & B
\end{tikzcd}$$

so $\tilde{\Lambda}$ is a homotopy from A^{γ_0} to A^{γ_1}

and $\tilde{\Lambda}\big|_{F_0 \times \{1\} \times [0,1]}$ is a homotopy

from $A_1^{\gamma_0}$ to $A_2^{\gamma_1}$

example: let (X, x_0) be a based topological space

set $P(X) = C(([0,1], \{0\}), (X, x_0))$

$= \{$ continuous maps $f : [0,1] \to X$ with $f(0) = x_0 \}$

and $p : P(X) \to X : \gamma \mapsto \gamma(1)$

lemma 5:

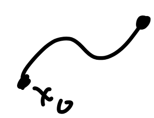
$$\boxed{\begin{array}{l} p : P(X) \to X \text{ is a fibration} \\ \text{and } P(X) \text{ is contractible} \end{array}}$$

Proof: we check the HLP

given
$$\begin{tikzcd} Y \times \{0\} \arrow[r, "f_0"] \arrow[d, hook] & P(X) \arrow[d, "p"] \\ Y \times [0,1] \arrow[r, "F"] & X \end{tikzcd}$$

then define $\tilde{F} : Y \times [0,1] \to P(X)$

by for $(y,s) \in Y \times [0,1]$,



$$\tilde{F}(y,s) : [0,1] \to X : t \mapsto \begin{cases} (f_0(y))\left(\frac{2t}{2-s}\right) & \text{for } t \in [0, \frac{2-s}{2}] \\ F(y, 2t-2+s) & \text{for } t \in [\frac{2-s}{2}, 1] \end{cases}$$

note: 1) well-defined since

$$f_0(y)\left(\frac{2(\frac{2-s}{2})}{2}\right) = f_0(y)(1) \text{ and}$$

$$F(y, 2(\tfrac{2-s}{2}) - 2 + s) = F(y, 0)$$

and since $p \circ f_0 = F$ these are same

2) $\tilde{F}(y,0)(t) = f_0(y)(t)$

3) $\tilde{F}(y,s)(0) = f_0(y)(0) = x_0$

4) $p \circ \tilde{F}(\gamma, s) = \tilde{F}(\gamma, s)(1) = F(\gamma, s)$

so $\tilde{F}$ a lift!

now $P(X)$ is contractible, since $[0,1]$ is

indeed

$$H : P(X) \times [0,1] \longrightarrow P(X)$$
$$(\gamma, s) \longmapsto \gamma((1-s)t)$$

is the strong deformation retraction to the constant path

<u>note</u>: $p^{-1}(x_0) = \Omega(X)$ the loop space of X

$\therefore p^{-1}(x) \simeq \Omega(X)$ for all $x \in X$ (if X is path connected)

so
$$\begin{array}{ccc} \Omega(X) & \longrightarrow & P(X) \\ & & \downarrow p \\ & & X \end{array}$$
is a fibration

<u>example</u>: given any continuous map $f : X \to Y$ earlier we saw f is homotopic to an inclusion.

Recall, if $C_f = (X \times [0,1]) \cup Y /_{(x,0) \sim f(x)}$ is mapping cylinder

then $Y \simeq C_f$ and

$$\begin{tikzcd} X \arrow[r, hook, "inc"'] \arrow[d, "id"'] & C_f \arrow[d, "\simeq"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

so upto homotopy $X \subset Y$

Now let $E = C(([0,1],\{0\}),(Y,X))$
$= $ paths in Y starting in X

$B = C(((\{0,1\},\{0\}),(Y,X))$
$= X \times Y$

exercise: $E \longrightarrow Y : \gamma \longmapsto \gamma(1)$ is a fibration
(proof very similar to proof of lemma 5)

note: $E \simeq X$ (just as we proved $P(X)$ is contractible)

$$\therefore \begin{tikzcd} X \arrow[r, phantom, "\simeq"] \arrow[d, hook] & E \arrow[d, "p"] \\ Y \arrow[r, phantom, "="] & Y \end{tikzcd}$$

so $f \simeq$ inclusion
$\simeq p$ a fibration

Slogan: $\boxed{\text{any map is a fibration (upto homotopy)}}$

lemma 6:

if $F \rightarrow E \xrightarrow{p} B$ is a fibration then $\pi_n(E,F) \cong \pi_n(B)$

Proof: let b_0 be the base point in B
$F = p^{-1}(b_0)$ and $e_0 \in F$ a base point

given $f:(D^n, \partial D^n) \to (E,F)$

then $p \circ f : (D^n, \partial D^n) \to (B, b_0)$

so p induces a map

$$p_* : \pi_n(E,F) \to \pi_n(B,b_0)$$

exercise: p_* is well-defined and a homomorphism

Claim: p_* is surjective

given $g:(D^n, \partial D^n) \to (B, b_0)$

think of D^n as $D^{n-1} \times [0,1]$

define $\tilde{g}_0 : D^{n-1} \times \{0\} \to E : x \mapsto e_0$

thinking of g as a homotopy of $p \circ \tilde{g}_0$

the HLP $\Rightarrow \exists$ a lift $\tilde{g} : D^{n-1} \times [0,1] \to E$ of g

and since $p \circ \tilde{g}(\partial(D^{n-1} \times [0,1])) = \{b_0\}$

$\tilde{g}(\partial(D^{n-1} \times [0,1])) \subset F = p^{-1}(b_0)$

$\therefore [\tilde{g}] \in \pi_n(E,F)$

clearly $p_*([\tilde{g}]) = [g]$ ✓

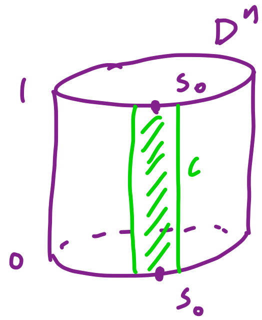
Claim: p_* is injective

suppose $f:(D^n, \partial D^n) \to (E, F)$
and $p_*([f]) = [0] \in \pi_n(B, b_0)$

i.e. $p \circ f \simeq$ constant b_0 map by the homotopy $H:(D^n, \partial D^n) \times [0,1] \to (B, b_0)$

so $H(x,0) = p \circ f(x)$, $H(x,1) = b_0$

and $H(\partial D^n \times [0,1]) = \{b_0\}$



base pt in ∂D^n

let $C = (\text{nbhd } s_0 \text{ in } \partial D^n) \times [0,1]$

and $A = (D^n \times \{0\}) \cup C$

as in proof of Th^m 4, $D^n \times [0,1] \cong A \times [0,1]$

so H is a map $A \times [0,1] \to B$

note: f on $D^n \times \{0\}$ and the constant map to e_0 is a lift of H on $A \times \{0\}$

so HLP $\Rightarrow \exists$ a lift $\tilde{H}: A \times [0,1] \to E$ of H and this gives

$$\tilde{H}:(D^n, \partial D^n) \times [0,1] \to E$$

that is a homotopy of f rel ∂D^n and rel s_0 to a map with image in F

$\therefore$ by lemma I.16, $[f]=0$ in $\pi_n(E,F)$ $\blacksquare$

Cor 7:

if $\begin{array}{c} F \to E \\ \downarrow p \\ B \end{array}$ is a fibration, then we get

a long exact sequence

$$\cdots \to \pi_n(F) \xrightarrow{i_*} \pi_n(E) \xrightarrow{p_*} \pi_n(B) \xrightarrow{\partial} \pi_{n-1}(F) \to \cdots$$

where i is inclusion and

$\pi_n(B) \cong \pi_n(E,F) \to \pi_{n-1}(F)$ is

from lemma 6 and Th$^{\underline{m}}$ I.17

Proof: Th$^{\underline{m}}$ I.17 gives

$$\cdots \to \pi_n(F) \to \pi_n(E) \to \pi_n(E,F) \to \pi_{n-1}(F) \to \cdots$$

now apply lemma 6 $\blacksquare$

Cor 8:

$\pi_k(S^{2n+1}) \cong \pi_k(\mathbb{C}P^n)$ for $k>2$

in particular $\pi_3(S^2) \cong \pi_3(S^3) \cong \mathbb{Z}$

$\mathbb{C}P^1$

Proof: recall we have the Hopf fibration

$$\begin{array}{c} S^1 \to S^{2n+1} \\ \qquad\quad \downarrow \\ \qquad\quad \mathbb{C}P^n \end{array}$$

so $\pi_k(S^1) \to \pi_k(S^{2n+1}) \to \pi_k(\mathbb{C}P^n) \to \pi_{k-1}(S^1)$

since $\mathbb{R}$ is the universal cover of S^1

$$\pi_k(S^1) = \pi_k(\mathbb{R}) = 0 \quad \forall k > 1$$

$\therefore$ if $k > 2$ then $k-1 > 1$ and we get

$$0 \to \pi_k(S^{2n+1}) \to \pi_k(\mathbb{C}P^n) \to 0$$

and $\therefore$ $\pi_k(S^{2n+1}) \cong \pi_k(\mathbb{C}P^n)$ ▦

note: also have

$$\begin{array}{ccccccc} \pi_2(S^3) & \to & \pi_2(S^2) & \to & \pi_1(S^1) & \to & \pi_1(S^3) \\ \| & & & & \cong & & \| \\ 0 & & & & \mathbb{Z} & & 0 \end{array}$$

so $\pi_2(S^2) \cong \mathbb{Z}$ without using Hurewicz

Cor 9:

$$\boxed{\begin{array}{l} X \text{ path connected then} \\ \qquad \pi_k(X) \cong \pi_{k-1}(\Omega X) \\ \qquad\qquad\qquad\qquad \text{loop space} \end{array}}$$

Remark: we already know this from Cor I. 8
but this is a different way to see it

Proof: recall above we constructed the fibration

$$\begin{array}{ccc} \Omega(X) & \to & P(X) \\ & & \downarrow \\ & & X \end{array}$$

and $P(X)$ contractible so

$$\underset{\overset{\parallel}{0}}{\pi_k(P(X))} \to \pi_k(X) \to \pi_{k-1}(\Omega X) \to \underset{\overset{\parallel}{0}}{\pi_{k-1}(P(X))}$$

so $\pi_k(X) \cong \pi_{k-1}(\Omega X)$ ▨

Cor 10:

$$\boxed{\begin{array}{ll} \pi_k(O(n-1)) \cong \pi_k(O(n)) & \text{for } k < n-2 \\ \pi_k(U(n)) \cong \pi_k(U(n-1)) & \text{for } k < 2n-2 \end{array}}$$

Proof: recall $V_{n,k}$ = k-frames in $\mathbb{R}^n$

$$\cong O(n)/O(n-k)$$

and $V_{n,1} \cong S^{n-1}$

so
$$\begin{array}{ccc} O(n-1) & \to & O(n) \\ & & \downarrow \\ & & S^{n-1} \end{array}$$
is a fiber bundle

$$\therefore \pi_{k+1}(S^{n-1}) \to \pi_k(O(n-1)) \to \pi_k(O(n)) \to \pi_k(S^{n-1})$$

if $k < n-2$ than $k+1 < n-1$ so

$$\pi_k(O(n-1)) \cong \pi_k(O(n))$$

similarly $V_{n,k}(\mathbb{C})$ = complex k-frams in $\mathbb{C}^n$

$$= U(n)/U(n-k)$$

so
$$\begin{array}{ccc} U(n-1) & \to & U(n) \\ & & \downarrow \\ & & S^{2n-1} \end{array}$$

gives second result ▨

Cor 10 $\Rightarrow$ for large n, $\pi_k(O(n))$ is independent of k small

note: we have inclusions

$$O(1) \hookrightarrow O(2) \hookrightarrow O(3) \hookrightarrow \dots \hookrightarrow O(k) \hookrightarrow \dots$$
$$A \longmapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \mapsto \dots$$

let $O = \lim_{n \to \infty} O(n) = \bigcup_{n=1}^{\infty} O(n)$

$U = \lim_{n \to \infty} U(n) = \bigcup_{n=1}^{\infty} U(n)$

easy to see $\pi_k(O) = \lim_{n \to \infty} \pi_k(O(n))$ constant at some point

so $\pi_k(O) \cong \pi_k(O(n))$ for $n > k+2$

$\pi_k(U) \cong \pi_k(U(n))$ for $n > \frac{k+2}{2}$

Big Theorem (Bott Periodicity)

$$\boxed{\begin{aligned} \pi_k(O) &\cong \pi_{k+8}(O) \\ \pi_k(U) &\cong \pi_{k+2}(U) \end{aligned}}$$

one can show:

k	0	1	2	3	4	5	6	7
$\pi_k(O)$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	0	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$

$$\pi_k(U) = \begin{cases} 0 & k \text{ even} \\ \mathbb{Z} & k \text{ odd} \end{cases}$$

note:

$$\begin{array}{ccc} SO(n) & \longrightarrow & O(n) \\ & & \downarrow \\ & & \{\pm 1\} \end{array}$$
is a bundle

so long exact sequence in π_k says
$$\pi_k(SO(n)) \cong \pi_k(O(n)) \quad \forall k > 0$$

$$\begin{array}{ccc} SU(n) & \longrightarrow & U(n) \\ & & \downarrow \\ & & S^1 \end{array}$$
is a bundle

so get $\pi_k(SU(n)) \cong \pi_k(U(n)) \ \forall k > 1$

Cor 11:

$$\pi_j(V_{n,k}(\mathbb{R})) = \begin{cases} 0 & \text{for } j < n-k \\ \mathbb{Z} & j = n-k \text{ even or } k=1 \\ \mathbb{Z}/2 & j = n-k \text{ odd} \end{cases}$$

$$\pi_j(V_{n,k}(\mathbb{C})) = \begin{cases} 0 & \text{for } j \le 2(n-k) \\ \mathbb{Z} & j = 2(n-k)+1 \end{cases}$$

Sketch of Proof:

recall $V_{n+1,k+1} = \dfrac{O(n+1)}{O(n-k)} = \dfrac{SO(n+1)}{SO(n-k)}$

so $V_{n,k} = \dfrac{SO(n)}{SO(n-k)} \subset V_{n+1,k+1}$

and
$$\begin{tikzcd} V_{n,k} \arrow[r] & V_{n+1,k+1} \arrow[d] \\ & SO(n+1)/SO(n) \cong S^n = V_{n+1,1} \end{tikzcd}$$

start with $k=1$:

$$\begin{tikzcd} S^{n-1} \arrow[r] & V_{n+1,2} \arrow[d, "p"] \\ & S^n \end{tikzcd}$$

so $\pi_j(S^n) \xrightarrow{\partial} \pi_{j-1}(S^{n-1}) \to \pi_{j-1}(V_{n+1,2}) \to \pi_{j-1}(S^n)$

if $j \le n-1$, then $\pi_j(S^n) = 0 = \pi_{j-1}(S^n)$

so $\pi_{j-1}(V_{n+1,2}) = \pi_{j-1}(S^{n-1}) = 0$ ✓

for $j=n$ we get

$$\begin{tikzcd} \pi_n(S^n) \arrow[r, "\partial"] \arrow[d, phantom, "\cong" rotate=-90] & \pi_{n-1}(S^{n-1}) \arrow[r] \arrow[d, phantom, "\cong" rotate=-90] & \pi_{n-1}(V_{n+1,2}) \arrow[r] & 0 \\ \mathbb{Z} & \mathbb{Z} & & \end{tikzcd}$$

so $\pi_{n-1}(V_{n+1,2}) \cong \pi_{n-1}(S^{n-1}) / \operatorname{im} \partial$

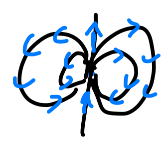
recall ∂ is defined by taking $f:(D^n, \partial D^n) \to (S^n, s_0)$

lifting to get $\tilde{f}:(D^n, \partial D^n) \to V_{n+1,2}$

and taking $\tilde{f}|_{\partial D^n} : \partial D^n \to S^{n-1}$

Facts: 1) $\exists$ a vector field v on S^n with a single zero at s_0, its index is $\begin{cases} 0 & n \text{ odd} \\ 2 & n \text{ even} \end{cases}$

e.g. $n=1$ 

$n=2$ 

2) if $f:(D^n, \partial D^n) \to S^n$ is the quotient map $D^n \to S^n$

(so f generates $\pi_n(S^n)$)

then $\tilde{f} : S^n \setminus \{s_0\} \to V_{n+1,2}$

$$\begin{array}{ccc} S^n \setminus \{s_0\} & \to & V_{n+1,2} \\ \| & & \rotatebox{90}{$\in$} \\ D^n & & \\ \rotatebox{90}{$\in$} & & \\ x & \longmapsto & \left(x, \dfrac{v(x)}{|v(x)|}\right) \end{array}$$

unit vector in $\mathbb{R}^n$

unit vector in $\mathbb{R}^n$ orthogonal to x

note: $p \circ \tilde{f} = f$ (p is project to first coordinate)

3) index of v is the degree of $\tilde{f}|_{\partial D} : \partial D \to S^{n-1}$

so $\partial[f] = \deg(f)[g]$ where g is a generator of $\pi_{n-1}(S^{n-1})$

$$\therefore \pi_{n-1}(V_{n+1,1}) = \begin{cases} \mathbb{Z} & n = n-k \text{ odd} \\ \mathbb{Z}/2 & n = n-k \text{ even} \end{cases}$$

so result true for $k=1$

now induct on k: assume result is true for for k

now $\pi_{j+1}(S^n) \to \pi_j(V_{n,k}) \to \pi_j(V_{n+1,k+1}) \to \pi_j(S^n)$

for $j < n-k$ we know $\pi_j(V_{n,k}) = 0$ so $\pi_j(V_{n+1,k+1}) = 0$

and for $j = n-k$ $\pi_j(V_{n+1,k+1}) \cong \pi_j(V_{n,k}) \cong \begin{cases} \mathbb{Z} & n-k \text{ odd} \\ \mathbb{Z}/2 & n-k \text{ even} \end{cases}$

$V_{n,k}(\mathbb{C})$ similar